ON THE BIFURCATION OF THE EQUILIBRIUM STATE OF A THREE-DIMENSIONAL ISOTROPIC ELASTIC SOLID UNDER LARGE SUBCRITICAL DEFORMATIONS

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General questions and the derivation of three-dimensional linearized equations for arbitrary subcritical deformations (linearized equilibrium equations in stresses) have been examined in a number of papers [1–11]. Particular forms of the elastic potential [3, 9, 12–16] have been applied in the derivation of the linearized equations in displacements for arbitrary subcritical deformations and in the solution of problems for a three-dimensional elastic isotropic solid in the majority of papers. For some of these problems, general solutions have been constructed in the case of homogeneous subcritical states [3, 14]. The linearized equations in displacements for an arbitrary form of the elastic potential have been obtained in [8], and a number of problems have been considered in such an approach [8, 17, 18].

Following [4], three-dimensional linearized equations in stresses are examined below, where components of the Green's finite strain tensor are chosen as strain characteristics. Linearized equations in displacements are obtained for an arbitrary form of the elastic potential. In the particular case of homogeneous subcritical states, general solutions are constructed which have been expressed in terms of the solution of second order equations for a solid with arbitrary cross-sectional outline. Proceeding from the general solutions, characteristic equations are obtained for a number of problems for an arbitrary form of the elastic potential. Certain forms of the elastic potential utilized in the literature are presented. Numerical examples are considered.

1. Formulation of the problem and method of investigation.

The equilibrium equations for arbitrary strains in Lagrangean coordinates which coincide with the Cartesian coordinates before deformation are [4]

$$[\sigma_{in}^* (\delta_{mn} + u_{m,n})]_{,i} = 0 \quad (i, m = 1, 2, 3) \tag{1.1}$$

We can write the boundary conditions as [4]

$$N_i \sigma_{in}^* (\delta_{mn} + u_{m,n}) = P_m^* \tag{1.2}$$

Here the N_i are the normal directions to the body surface before the strain, P_m^* the components of the external force vector along the axes of the Cartesian coordinate system, which act at the same point of the body surface after deformation, but are referred to unit area before deformation. The other quantities can be represented as follows:

$$\sigma_{ij}^* = \frac{S_i^*}{S_i} \frac{\sigma_{ij}}{1+E_j}, \qquad \frac{S_i^*}{S_i} = \sqrt{(1+2\varepsilon_{nn})(1+2\varepsilon_{mm}) - (2\varepsilon_{nm})^2}$$
$$1 + E_j = \sqrt{1+2\varepsilon_{jj}}, \qquad 2\varepsilon_{ij} = u_{i,j} + u_{j,i} + u_{s,i} u_{s,j} \tag{1.3}$$

In (1.3) there is no summation over i, j, n, m; i, n, m are an even commutation of 1, 2, 3; σ_{ij}^* are generalized stresses [4], σ_{ij} are the physical components of the stress tensor, E_j are the elongations, S^*_N / S_N is the change in area of the areas determined by the direction N, ε_{ij} are components of the finite strain tensor.

According to [2, 19, 4], the generalized stresses σ_{ij}^* are defined in terms of the elastic potential Φ as follows:

$$\sigma_{ij}^* = \frac{1}{2}\left(\frac{\partial}{\partial\varepsilon_{ij}} + \frac{\partial}{\partial\varepsilon_{ji}}\right)\Phi \tag{1.4}$$

Let us assume that the elastic potential is a twice continuously differentiable function of three independent algebraic invariants

$$\Phi = \Phi(I_1, I_2, I_3), \quad I_1 = \varepsilon_{nn}, \quad I_2 = \varepsilon_{nm}\varepsilon_{mn}, \quad I_3 = \varepsilon_{np}\varepsilon_{pm}\varepsilon_{mn} \tag{1.5}$$

Hence we can write

$$\sigma_{ij}^* = \psi_0(I_1, I_2, I_3)\,\delta_{ij} + \psi_1(I_1, I_2, I_3)\,\varepsilon_{ij} + \psi_2(I_1, I_2, I_3)\,\varepsilon_{in}\varepsilon_{nj}$$
$$\psi_0 = \partial\Phi/\partial I_1, \quad \psi_1 = 2\partial\Phi/\partial I_2, \quad \psi_2 = 3\partial\Phi/\partial I_3 \tag{1.6}$$

The first expression in (1.6) can be obtained from the relationships between two coaxial, symmetric tensors of the second rank [20].

Following [4], let us linearize the relationships (1.1)–(1.6). To do this we represent all the quantities as

$$\sigma_{in}^{*\prime} = \sigma_{in}^{0*\circ} + \sigma_{in}^*, \quad u_i' = u_i^\circ + u_i, \quad \varepsilon_{ij}' = \varepsilon_{ij}^\circ + \varepsilon_{ij} \tag{1.7}$$
$$I_k' = I_k^\circ + I_k, \ldots, \; P_m^{*\prime} = P_m^{*\circ} + P_m^*$$

The superscript $^\circ$ here denotes all quantities in the unperturbed state determined from the relationships (1.1)–(1.6), while the perturbations are unmarked.

As a result of linearization we obtain

$$[\sigma_{in}^*(\delta_{nm} + u^\circ_{m,n}) + \sigma_{in}^{*\circ}u_{m,n}]_{,i} = 0 \tag{1.8}$$

and the boundary conditions

$$N_i[\sigma_{in}^*(\delta_{nm} + u^\circ_{m,n}) + \sigma_{in}^{*\circ}u_{m,n}] = P_m^* \tag{1.9}$$

The linearized relationships (1.6) become

$$\sigma_{ij}^* = I_k A_{kij} + B\varepsilon_{ij} + C_{in}\varepsilon_{nj} + D_{nj}\varepsilon_{in} \tag{1.10}$$

wherein the following notation has been introduced

$$A_{kij} = \delta_{ij}\frac{\partial\psi_0^\circ}{\partial I_k^\circ} + \varepsilon_{ij}^\circ\frac{\partial\psi_1^\circ}{\partial I_k^\circ} + \overset{\circ}{\varepsilon}_{in}\overset{\circ}{\varepsilon}_{nj}\frac{\partial\psi_2^\circ}{\partial I_k^\circ}, \quad C_{in} = \varepsilon_{in}^\circ\psi_2^\circ$$
$$D_{nj} = \varepsilon_{nj}^\circ\psi_2^\circ, \quad B = \psi_1^\circ, \quad I_1 = \varepsilon_{nn}, \quad I_2 = 2\overset{\circ}{\varepsilon}_{nm}\varepsilon_{mn} \tag{1.11}$$
$$I_3 = 3\varepsilon_{nm}\overset{\circ}{\varepsilon}_{np}\overset{\bullet}{\varepsilon}_{pm}, \quad 2\varepsilon_{ij} = u_{i,j} + u_{j,i} + \overset{\circ}{u}_{s,i}u_{s,j} + u_{s,i}\overset{\bullet}{u}_{s,j}$$

Substituting (1.10) and (1.11) into (1.8) and (1.9), we obtain the equations of boundary conditions for the problem of bifurcation of the equilibrium state of a three-dimensional elastic isotropic solid under arbitrary subcritical strains for an arbitrary form of the elastic potential, hence it is necessary to take account of the representation of the quantities ψ_i in the terms of the elastic potential (1.6).

It must be noted that the generalized stresses $\sigma_{in}^{*\circ}$ are defined on the body surface in terms of $P_m^{*\circ}$ as follows:

$$N_i\sigma_{in}^{*\circ}(\delta_{mn} + \overset{\circ}{u}_{m,n}) = P_m^{*\circ} \tag{1.12}$$

Let us turn to an analysis of the homogeneous subcritical states.

2. General solutions for homogeneous subcritical states. Let us examine the case when the subcritical state is defined by the following displacements:

$$u_m^\circ = \delta_{im}(\lambda_i - 1)x_i, \quad E_i^\circ = \lambda_i - 1, \quad \lambda_i = \text{const} \tag{2.1}$$

It follows from (1.6) and (2.1) that the subcritical state is triaxial

$$\sigma_{mm}^{*\circ}\lambda_m = P_m^{*\circ}, \quad \sigma_{im}^{*\circ} = 0, \quad i \neq m \tag{2.2}$$

To determine the quantities $\sigma_{mm}^{*\circ}$ from (1.6) we obtain the expression

$$\sigma_{mm}^{*\circ} = \frac{\partial\Phi^\circ}{\partial I_1^\circ} + (\lambda_m^2 - 1)\frac{\partial\Phi^\circ}{\partial I_2^\circ} + 3\left(\frac{\lambda_m^2 - 1}{2}\right)^2\frac{\partial\Phi^\circ}{\partial I_3^\circ} \tag{2.3}$$

There is no summation over m in the relationships (2.2) and (2.3).

According to (2.1), (1.3) and (1.5), we obtain values of the quantities of the subcritical state, and according to (2.1) and (1.11), values of the perturbations

$$2\varepsilon_{ij}^\circ = \delta_{ij}(\lambda_i^2 - 1), \quad I_1^\circ = {}^1/_2(\lambda_n\lambda_n - 3), \quad I_2^\circ = {}^1/_4(\lambda_n^2 - 1)(\lambda_n^2 - 1) \tag{2.4}$$

$$I_3^\circ = {}^1/_8(\lambda_n^2 - 1)(\lambda_n^2 - 1)(\lambda_n^2 - 1)$$

$$2\varepsilon_{ij} = \lambda_i u_{i,\,j} + \lambda_j u_{j,\,i}, \quad I_1 = \lambda_n u_{n,\,n}, \quad I_2 = (\lambda_n^2 - 1)\lambda_n u_{n,\,n} \tag{2.5}$$

$$I_3 = {}^3/_4(\lambda_n^2 - 1)^2\lambda_n u_{n,\,n}, \qquad n = 1, 2, 3$$

Taking account of (2.1) and (2.2), the Eqs. (1.8) and the boundary conditions (1.9) take the following form:

$$[\sigma_{im}^*\lambda_m + \sigma_{ii}^{*\circ}u_{m,\,i}]_{,\,i} = 0 \tag{2.6}$$

$$N_i[\sigma_{im}^*\lambda_m + \sigma_{ii}^{*\circ}u_{m,\,i}] = P_m^* \tag{2.7}$$

Taking account of (1.6), (1.10), (1.11), (2.1), (2.4) and (2.5), we obtain

$$\sigma_{ij}^* = \delta_{ij}a_{ik}\lambda_k u_{k,\,k} + (1 - \delta_{ij})G_{ij}(\lambda_i u_{i,\,j} + \lambda_j u_{j,\,i}) \tag{2.8}$$

wherein we have introduced the notation

$$a_{ik} = \left[\frac{\partial^2\Phi^\circ}{\partial I_1^\circ\partial I_t^\circ} + (\lambda_i^2 - 1)\frac{\partial^2\Phi^\circ}{\partial I_2^\circ\partial I_t^\circ} + 3\left(\frac{\lambda_i^2 - 1}{2}\right)^2\frac{\partial^2\Phi^\circ}{\partial I_3^\circ\partial I_t^\circ}\right] \times$$

$$\times\left[\delta_{t1} + (\lambda_k^2 - 1)\delta_{t2} + 3\left(\frac{\lambda_k^2 - 1}{2}\right)^2\delta_{t3}\right] + \delta_{ik}\left[2\frac{\partial\Phi^\circ}{\partial I_2^\circ} + 3(\lambda_k^2 - 1)\frac{\partial\Phi^\circ}{\partial I_3^\circ}\right],$$

$$G_{ij} = \frac{1}{2}\left[2\frac{\partial\Phi^\circ}{\partial I_2^\circ} + 3\frac{\lambda_i^2 + \lambda_j^2 - 2}{2}\frac{\partial\Phi^\circ}{\partial I_3^\circ}\right] \tag{2.9}$$

It follows from (2.8) and (2.9) that the matrix $\|b_{ik}\|$ $(b_{ik} = a_{ik}\lambda_k)$ is not symmetric in the general case; therefore, the relationships between the increments (perturbations) of the generalized stresses and the derivatives of the increments (perturbations) of the displacements do not agree with the corresponding Hooke's law relationships for an orthotropic body under small deformations.

Substituting (2.8) into (2.6), we obtain the fundamental equations in displacements

$$L_{mj}u_j = 0 \qquad (j, m = 1, 2, 3) \tag{2.10}$$

where

$$L_{mj} = a_{mj}\lambda_j\frac{\partial}{\partial x_m\partial x_j} + (1 - \delta_{jm})G_{jm}\lambda_j\frac{\partial^2}{\partial x_m\partial x_j} +$$

$$+ (1 - \delta_{im})G_{im}\delta_{jm}\lambda_j\frac{\partial^2}{\partial x_i^2} + \frac{\sigma_{ii}^{*\circ}}{\lambda_j}\delta_{jm}\frac{\partial^2}{\partial x_i^2} \tag{2.11}$$

We represent the solution of the system of equations (2.10) in one of three forms, or as their linear combination

$$u_i^{(j)} = \frac{\partial \det \| L_{rs} \|}{\partial (L_{ji})} \Phi^{(j)} \qquad (i, j = 1, 2, 3) \tag{2.12}$$

There is no summation over j in (2.12).

The functions $\Phi^{(j)}$ are determined from the equations

$$\det \| L_{rs} \| \, \Phi^{(j)} = 0 \tag{2.13}$$

Substituting (2.8) into the boundary conditions (2.7), we derive the boundary conditions in displacements, but do not present them.

The quantity $P^\circ{}_m (S_m{}^* \,/\, S_m)^\circ$ (not summed over m) can be considered as the intensity of the surface loading referred to unit area in the body before deformation. Let it be denoted by $T_{mm}{}^\circ$. We hence obtain

$$\overset{\circ}{T}_{mm} = \overset{*\circ}{\sigma}_{mm} \lambda_m \qquad \text{(not summed over } m\text{)} \tag{2.14}$$

The relationships (2.3) and (2.14) are useful in solving particular problems.

Let us investigate the case when the equalities

$$\overset{*\circ}{\sigma}_{11} = \overset{*\circ}{\sigma}_{22} \neq 0, \qquad \overset{*\circ}{\sigma}_{33} \neq 0 \tag{2.15}$$

are satisfied, or what is identical

$$\lambda_1 = \lambda_2 \neq 0, \qquad \lambda_3 \neq 0 \tag{2.16}$$

We see from (2.9) that the equalities

$$a_{11} \equiv a_{22}, \quad a_{13} \equiv a_{23}, \quad 2G_{12} = a_{11} - a_{12}, \quad a_{ij} \equiv a_{ji} \tag{2.17}$$

are then satisfied.

Taking (2.17) into account, we represent the solution of (2.13) as

$$\Phi^{(j)} = \Phi_1 + \Phi_2 + \Phi_3 \qquad \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \zeta_i{}^2 \frac{\partial^2}{\partial x_3^2}\right) \Phi_i = 0 \quad \text{(not summed over } i\text{)} \tag{2.18}$$

The quantities $\zeta^2{}_i$ are roots of a cubic equation, which we do not present because of its awkwardness. We obtain the following values of the roots as a result of solving the equation:

$$\zeta_1{}^2 = \frac{G_{13} + \sigma_{33}{}^{*\circ}\lambda_1{}^{-2}}{G_{12} + \sigma_{11}{}^{*\circ}\lambda_1{}^{-2}}, \quad \zeta_{2,3}{}^2 = C \pm \left[C^2 - \frac{(a_{33} + \sigma_{33}{}^{*\circ}\lambda_3{}^{-2})(G_{13} + \sigma_{33}{}^{*\circ}\lambda_1{}^{-2})}{(a_{11} + \sigma_{11}{}^{*\circ}\lambda_1{}^{-2})(G_{13} + \sigma_{11}{}^{*\circ}\lambda_3{}^{-2})}\right]^{1/2}$$

$$2C = \frac{a_{33} + \sigma_{33}{}^{*\circ}\lambda_3{}^{-2}}{G_{13} + \sigma_{11}{}^{\circ}\lambda_3{}^{-2}} + \frac{G_{13} + \sigma_{33}{}^{*\circ}\lambda_1{}^{-2}}{a_{11} + \sigma_{11}{}^{*\circ}\lambda_1{}^{-2}} - \frac{(a_{13} + G_{13})^2}{(a_{11} + \sigma_{11}{}^{*\circ}\lambda_1{}^{-2})(G_{13} + \sigma_{11}{}^{*\circ}\lambda_3{}^{-2})} \tag{2.19}$$

Let us proceed as follows to determine the displacements. Let us set

$$\Phi^{(1)} = \Phi_1, \quad \Phi^{(2)} = \Phi_1, \quad \Phi^{(3)} = \Phi_2 + \Phi_3, \quad u_i = u_i^{(1)} - u_i^{(2)} + u_i^{(3)} \tag{2.20}$$

into (2.12) and (2.18), and let us introduce functions Ψ_i, which are solutions of (2.18) and are connected with the functions Φ_i by the relationships

$$\Psi_1 = (a_{12} + G_{12})(G_{13} + \sigma_{11}{}^{*\circ}\lambda_3{}^{-2})\lambda_1\lambda_3 \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\left\{\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \right.$$

$$\left. + \left[\frac{a_{33} + \sigma_{33}{}^{*\circ}\lambda_3{}^{-2}}{G_{13} + \sigma_{11}{}^{*\circ}\lambda_3{}^{-2}} - \frac{(a_{13} + G_{13})^2}{(a_{12} + G_{12})(G_{13} + \sigma_{11}{}^{*\circ}\lambda_3{}^{-2})}\right]\frac{\partial^2}{\partial x_3^2}\right\}\Phi_1 \tag{2.21}$$

$$\Psi_i = (a_{13} + G_{13})(G_{12} + \sigma_{11}{}^{*\circ}\lambda_1{}^{-2})\lambda_1\lambda_3 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \zeta_1{}^2 \frac{\partial^2}{\partial x_3^2}\right)\Phi_i \qquad (i = 2, 3)$$

We hence obtain from the expressions (2.11), (2.12), (2.17)–(2.21)

$$u_1 = \frac{\partial}{\partial x_2}\Psi_1 - \frac{\partial^2}{\partial x_1 \partial x_3}(\Psi_2 + \Psi_3), \quad u_2 = -\frac{\partial}{\partial x_1}\Psi_1 - \frac{\partial^2}{\partial x_2 \partial x_3}(\Psi_2 + \Psi_3) \tag{2.22}$$

$$u_3 = \frac{\lambda_1}{\lambda_3}\frac{a_{11} + \sigma_{11}^{*\circ}\lambda_1^{-2}}{a_{13} + G_{13}}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{G_{13} + \sigma_{33}^{*\circ}\lambda_1^{-2}}{a_{11} + \sigma_{11}^{*\circ}\lambda_1^{-2}}\frac{\partial^2}{\partial x_3^2}\right)(\Psi_2 + \Psi_3)$$

The general solutions (2.22) for a curved cross-sectional outline become

$$u_n = \frac{\partial}{\partial s}\Psi_1 - \frac{\partial^2}{\partial n \partial x_3}(\Psi_2 + \Psi_3), \quad u_s = -\frac{\partial}{\partial n}\Psi_1 - \frac{\partial^2}{\partial s \partial x_3}(\Psi_2 + \Psi_3) \tag{2.23}$$

$$u_3 = \frac{\lambda_1}{\lambda_3}\frac{a_{11} + \sigma_{11}^{*\circ}\lambda_1^{-2}}{a_{13} + G_{13}}\frac{\partial^2}{\partial x_3^2}\left[\left(\frac{G_{13} + \sigma_{33}^{*\circ}\lambda_1^{-2}}{a_{11} + \sigma_{11}^{*\circ}\lambda_1^{-2}} - \zeta_2^2\right)\Psi_2 + \right.$$
$$\left. + \left(\frac{G_{13} + \sigma_{33}^{*\circ}\lambda_1^{-2}}{a_{11} + \sigma_{11}^{*\circ}\lambda_1^{-2}} - \zeta_3^2\right)\Psi_3\right]$$

Here u_n is the displacement along the normal, and u_s the displacement along the tangent.

Let us write the boundary conditions on a cylindrical surface for a curved outline when the Ox_3-axis coincides with the cylinder axis. After a number of manipulations we obtain from (2.7)

$$\lambda_1^2(a_{11} + \sigma_{11}^{*\circ}\lambda_1^{-2})(u_{n,n} - u_1 N_{1,n} - u_2 N_{2,n}) + \lambda_1^2 a_{12}(u_{s,s} - u_2 N_{1,s} + u_1 N_{2,s}) +$$
$$+ \lambda_1\lambda_3 a_{13} u_{3,3} = P_n^*, \quad \lambda_1^2 G_{12}[(1 + \sigma_{11}^{*\circ}\lambda_1^{-2}G_{12}^{-1})(u_{s,n} + u_1 N_{2,n} - \tag{2.24}$$
$$- u_2 N_{1,n}) + u_{n,s} - u_1 N_{1,s} - u_2 N_{2,s}] = P_s^*, \quad G_{13}[\lambda_1\lambda_3 u_{n,3} + \lambda_3^2(1 +$$
$$+ \sigma_{11}^{*\circ}\lambda_3^{-2}G_{13}^{-1})u_{3,n}] = P_3^*, \; P_n^* = P_1^* N_1 + P_2^* N_2, \; P_s^* = -P_1^* N_2 + P_2^* N_1$$

For a circular cylindrical coordinate system, the boundary conditions (2.24) become after a number of manipulations

$$\lambda_1^2(a_{11} + \sigma_{11}^{*\circ}\lambda_1^{-2})u_{r,r} + \lambda_1^2 a_{12}(r^{-1}u_{\theta,\theta} + r^{-1}u_r) + \lambda_1\lambda_3 a_{13}u_{3,3} = P_r^*$$
$$\lambda_1^2 G_{12}[(1 + \sigma_{11}^{*\circ}\lambda_1^{-2}G_{12}^{-1})u_{\theta,r} + r^{-1}u_{r,\theta} - r^{-1}u_\theta] = P_\theta^* \tag{2.25}$$
$$G_{13}[\lambda_1\lambda_3 u_{r,3} + \lambda_3^2(1 + \sigma_{11}^{*\circ}\lambda_3^{-2}G_{13}^{-1})u_{3,r}] = P_3^*$$

Let us write the boundary conditions for $x_3 = \text{const}$. From (2.7) we obtain

$$G_{13}[\lambda_1^2(1 + \sigma_{33}^{*\circ}\lambda_1^{-2}G_{13}^{-1})u_{n,3} + \lambda_1\lambda_3 u_{3,n}] = P_n^*$$
$$G_{23}[\lambda_1^2(1 + \sigma_{33}^{*\circ}\lambda_1^{-2}G_{23}^{-1})u_{s,3} + \lambda_1\lambda_3 u_{3,s}] = P_s^* \tag{2.26}$$
$$a_{13}\lambda_1\lambda_3 \operatorname{div}\mathbf{u} + \lambda_3^2(a_{33} - a_{13}\lambda_1\lambda_3^{-1} + \sigma_{33}^{*\circ}\lambda_3^{-2})u_{3,3} = P_3^*$$

It must be noted that the quantities P_n^*, P_s^*, P_3^* in condition (2.26) do not agree with the analogous quantities in condition (2.24).

Let us examine the case of plane strain in the x_1x_3 plane, when $\sigma_{33}^{*\circ} \neq 0$, and $\sigma_{11}^{*\circ} = 0$. In this case, we have the following relationships for the perturbations:

$$\sigma_{11}^* = a_{11}\lambda_1 u_{1,1} + a_{13}\lambda_3 u_{3,3}, \quad \sigma_{13}^* = G_{13}(\lambda_1 u_{1,3} + \lambda_3 u_{3,1}) \tag{2.27}$$
$$\sigma_{33}^* = a_{31}\lambda_1 u_{1,1} + a_{33}\lambda_3 u_{3,3}, \quad \sigma_{13}^* \equiv \sigma_{31}^*$$

The displacements are determined as follows:

$$u_1 = \left[G_{13}\lambda_3\frac{\partial^2}{\partial x_1^2} + (a_{33} + \sigma_{33}^{*\circ}\lambda_3^{-2})\lambda_3\frac{\partial^2}{\partial x_3^2}\right](\Psi_2 + \Psi_3)$$
$$u_3 = -\lambda_1(a_{13} + G_{13})\frac{\partial^2(\Psi_2 + \Psi_3)}{\partial x_1 \partial x_3} \tag{2.28}$$

The functions Ψ_i are solutions of the equations

$$\left(\frac{\partial^2}{\partial x_1^2} + \zeta_i^2 \frac{\partial^2}{\partial x_3^2}\right)\Psi_i = 0 \quad (i = 2, 3) \quad \text{(not summed over } i\text{)} \qquad (2.29)$$

The boundary conditions for $x_1 = \text{const}$ have the form

$$\sigma_{11}{}^*\lambda_1 = P_1{}^*, \qquad \sigma_{13}{}^*\lambda_3 = P_3{}^* \qquad (2.30)$$

We apply the results expounded in this Section to the solution of a number of problems. For these problems we shall obtain characteristic equations for the general form of the elastic potential.

The boundary conditions in displacements are written as is customary.

In evaluating the quantities a_{ij} and G_{ij} by means of (2.9) for plane strain it is necessary to introduce the simplifications inherent to plane strain.

3. Characteristic equations. Let us examine the stability of a cylindrical shell of thickness $2h$, length l and middle surface radius R for compression along the generator. In this case it is necessary to set $\sigma_{11}^{*\circ} = 0$ everywhere in (2.19), (2.23) and (2.25), and to consider $\sigma_{33}^{*\circ}$ negative. For the nonaxisymmetric buckling mode the solutions of (2.18) are selected as

$$\Psi_1 = [A_{mn}^{11} I_n(\gamma\zeta_1 r) + A_{mn}^{12} K_n(\gamma\zeta_1 r)] \sin\gamma x_3 \sin n\theta, \quad \gamma = m\pi/l \qquad (3.1)$$

$$\Psi_i = [A_{mn}^{i1} I_n(\gamma\zeta_i r) + A_{mn}^{i2} K_n(\gamma\zeta_i r)] \cos\gamma x_3 \cos n\theta \quad (i = 2,3)$$

When the solutions are selected in the form (3.1) on the shell endfaces, hinged support conditions are satisfied. In (3.1) and below, I_n and K_n denote the Bessel functions of pure imaginary argument and the Macdonald function [21]. Substituting (3.1) and (2.23) into the boundary conditions (2.25) for $P_r{}^* \equiv P_\theta{}^* \equiv P_3{}^* = 0$, as a result of the customary procedure, we obtain the characteristic equation in the form

$$\det \|\alpha_{ij}\| = 0 \qquad (i, j = 1, 2, \ldots, 6) \qquad (3.2)$$

where we have introduced the notation

$$\alpha_{11}(I_{n+1}, I_n) = \frac{2b_1 n\zeta_1}{\gamma(\varkappa+\varepsilon)} I_{n+1}[\zeta_1(\varkappa+\varepsilon)] + \frac{2b_1 n(n-1)}{\gamma(\varkappa+\varepsilon)^2} I_n[\zeta_1(\varkappa+\varepsilon)] \qquad (3.3)$$

$$\alpha_{13}(I_{n+1}, I_n, \zeta_2) = -\frac{2b_1\zeta_2}{\varkappa+\varepsilon} I_{n+1}[\zeta_2(\varkappa+\varepsilon)] +$$

$$+\left[\zeta_2{}^2 + k_2 + \frac{2b_1 n(n-1)}{(\varkappa+\varepsilon)^2}\right] I_n[\zeta_2(\varkappa+\varepsilon)]$$

$$\alpha_{12} \equiv \alpha_{11}(-K_{n+1}, K_n), \quad \alpha_{14} \equiv \alpha_{13}(-K_{n+1}, K_n, \zeta_2), \quad \alpha_{15} \equiv \alpha_{13}(I_{n+1}, I_n, \zeta_3)$$

$$\alpha_{16} \equiv \alpha_{13}(-K_{n+1} K_n, \zeta_3), \quad \alpha_{31}(I_n) = \frac{b_1 n}{\gamma(\varkappa+\varepsilon)} I_n[\zeta_1(\varkappa+\varepsilon)], \quad \alpha_{32} \equiv \alpha_{31}(K_n)$$

$$\alpha_{33}(I_{n+1}, I_n, \zeta_2) = (\zeta_2{}^3 + k_1\zeta_2) I_{n+1}[\zeta_2(\varkappa+\varepsilon)] + \frac{n(\zeta_2{}^2 + k_1)}{\varkappa+\varepsilon} I_n[\zeta_2(\varkappa+\varepsilon)]$$

$$\alpha_{53}(I_{n+1}, I_n, \zeta_2) = -\frac{2n\gamma\zeta_2}{\varkappa+\varepsilon} I_{n+1}[\zeta_2(\varkappa+\varepsilon)] + \frac{2n\gamma(1-n)}{(\varkappa+\varepsilon)^2} I_n[\zeta_2(\varkappa+\varepsilon)]$$

$$\alpha_{52} \equiv \alpha_{51}(-K_{n+1}, K_n), \qquad \alpha_{54} \equiv \alpha_{53}(-K_{n+1}, K_n, \zeta_2), \qquad \alpha_{55} \equiv \alpha_{53}(I_{n+1}, I_n, \zeta_3)$$

$$\alpha_{56} \equiv \alpha_{53}(-K_{n+1}, K_n, \zeta_3), \qquad \varkappa = m\pi R/l, \qquad \varepsilon = m\pi h/l$$

$$k_1 = (a_{13} - \sigma_{33}{}^{*\circ}\lambda_1{}^{-2})/a_{11}, \quad k_2 = a_{13}(G_{13} + \sigma_{33}{}^{*\circ}\lambda_1{}^{-2})/a_{11}G_{13},$$

$$b_1 = G_{12}(a_{13} + G_{13})/a_{11}$$

To determine the elements of the second, fourth, and sixth lines, it is necessary to reverse the sign before the ε, in the first, third and fifth lines, respectively. In the case of the axisymmetric buckling mode, we select the solutions of (2.18) in the form

$$\Psi_1 \equiv 0, \quad \Psi_i = [A_{m0}^{i1} I(\gamma\zeta_i r) + A_{m0}^{i2} K(\gamma\zeta_i r)] \cos \gamma x_3 \qquad (i = 2,3) \tag{3.4}$$

For $i, j = 1, 2, 3, 4$ the characteristic equation has the form (3.2).

Let us represent values of the elements of the determinant as

$$\begin{gathered}\alpha_{11}(I_1, I, \zeta_2) = -\frac{2b_1\zeta_2}{\varkappa + \varepsilon} I_1[\zeta_2(\varkappa + \varepsilon)] + (\zeta_2^2 + k_2) I[\zeta_2(\varkappa + \varepsilon)] \\ \alpha_{12} \equiv \alpha_{11}(-K_1, K, \zeta_2), \quad \alpha_{13} \equiv \alpha_{11}(I_1, I, \zeta_3), \quad \alpha_{14} \equiv \alpha_{11}(-K_1, K, \zeta_3) \\ \alpha_{31}(I_1, \zeta_2) = (\zeta_2^3 + k_1\zeta_2) I_1[\zeta_2(\varkappa + \varepsilon)], \quad \alpha_{32} \equiv \alpha_{31}(-K_1, \zeta_2) \\ \alpha_{33} \equiv \alpha_{31}(I_1, \zeta_3), \quad \alpha_{34} \equiv \alpha_{31}(-K_1, \zeta_3)\end{gathered} \tag{3.5}$$

To determine the elements of the second and fourth lines, the sign in front of the ε must be reversed in the elements of the first and third lines, respectively.

Let us consider the formation of an axisymmetric neck of length l in the tension of a cylindrical sample of radius R. To this end, we set $A_{m0}^{i2} = 0$ in the solution (3.4), and substitute (3.4) into the boundary conditions (2.25) for $P_r^* \equiv P_\theta^* \equiv P^*_3 = 0$. As a result of the customary procedure we obtain the characteristic equation in the form (3.2) for $i, j = 1, 2$. The elements of the determinant are the following:

$$\begin{gathered}\alpha_{11}(\zeta_2) = -\frac{2b_1\zeta_2}{\varkappa} I_1(\zeta_2\varkappa) + (\zeta_2^2 + k_2) I(\zeta_2\varkappa), \qquad \alpha_{12} \equiv \alpha_{11}(\zeta_3) \\ \alpha_{21}(\zeta_2) = \zeta_2(\zeta_2^2 + k_1) I_1(\zeta_2\varkappa), \qquad \alpha_{22} \equiv \alpha_{21}(\zeta_3)\end{gathered} \tag{3.6}$$

If the Macdonald functions replace the Bessel functions in (3.6), and the sign before the K_1 is reversed, then we obtain the elements of the characteristic determinant for the problem of surface instability of a cylindrical cavity under axisymmetric deformations. The length of the neck is found by minimizing the roots of (3.2) and (3.6) for $i, j = 1, 2$.

Let us examine the stability of a rod of length l of circular cross section with radius R under compression. Let us limit ourselves to the plane buckling mode for the hinge support case in the x_1x_3 plane. Let us select the solutions of the fundamental equations (2.18) as

$$\Psi_1 = A_m^{(1)} I_1(\gamma\zeta_1 r) \sin \gamma x_3 \sin \theta, \quad \Psi_i = A_m^{(i)} I_1(\gamma\zeta_i r) \cos \gamma x_3 \cos \theta \qquad (i = 2,3) \tag{3.7}$$

As a result of the customary procedure we obtain a characteristic equation in the form (3.2) for $i, j = 1, 2, 3$. Values of the elements of the determinant are

$$\begin{gathered}\alpha_{11} = (a_{11} - a_{12})\varkappa\zeta_1 I_2(\varkappa\zeta_1), \quad \alpha_{12}(\zeta_2) = \varkappa a_{11}(\varkappa\zeta_2)^2 I_1''(\varkappa\zeta_2) + a_{12}\varkappa^2\zeta_2 I_2(\varkappa\zeta_2) + \\ + \varkappa^3 a_{13}(a_{13} + G_{13})^{-1}(G_{13} + \sigma_{33}^{*\circ}\lambda_1^{-2} - a_{11}\zeta_2^2) I_1(\varkappa\zeta_2), \quad \alpha_{13} \equiv \alpha_{12}(\zeta_3) \\ \alpha_{21} = \varkappa\zeta_1 I_2(\varkappa\zeta_1) - (\varkappa\zeta_1)^2 I_1''(\varkappa\zeta_1), \quad \alpha_{22}(\zeta_2) = -2\varkappa^2\zeta_2 I_2(\varkappa\zeta_2) \\ \alpha_{23} \equiv \alpha_{22}(\zeta_3), \quad \alpha_{31} = I_1(\varkappa\zeta_1), \quad \alpha_{32}(\zeta_2) = \varkappa^2\zeta_2(a_{13} + G_{13})^{-1}(a_{13} - \\ - \sigma_{33}^{*\circ}\lambda_1^{-2} + \zeta_2^2 a_{11}) I_1'(\varkappa\zeta_2), \quad \alpha_{33} \equiv \alpha_{32}(\zeta_3)\end{gathered} \tag{3.8}$$

It must be noted that the quantities ζ_i^2 in all the expressions (3.3), (3.5), (3.6), (3.8) must be computed by means of (2.19) for $\sigma_{11}^{*\circ} = 0$, where $\sigma_{33}^{*\circ} < 0$ in (3.3), (3.5) and (3.8), and $\sigma_{33}^{*\circ} > 0$ in (3.6).

Let us examine the stability of a rigidly clamped circular plate of thickness $2h$ and radius R under uniform compression in the x_1x_2 plane. The solutions of the fundamental equations (2.18) are selected in the form

$$\Psi_1 = A_{kn}^{(i)} J_n \left(\frac{\varkappa_k}{R} r \right) \operatorname{sh} \frac{\varkappa_k}{R\zeta_1} x_3 \sin n\theta, \qquad J_n'(x_k) = 0 \tag{3.9}$$

$$\Psi_i = A_{kn}^{(i)} J_n \left(\frac{\varkappa_k}{R} r \right) \operatorname{ch} \frac{\varkappa_k}{R\zeta_i} x_3 \cos n\theta, \qquad i = 2,3$$

It is necessary to note that the boundary conditions are hence satisfied in the integral sense, and $\sigma_{33}^{*\circ} = 0$. For the axisymmetric buckling mode the solution becomes

$$\Psi_1 = 0 \qquad \Psi_i = A_{k0}^{(i)} J \left(\frac{\varkappa_k}{R} r \right) \operatorname{ch} \frac{\varkappa_k}{R\zeta_i} x_3 \qquad (i = 2,3) \tag{3.10}$$

Substituting (3.10), (2.19), (2.23) into the boundary conditions (2.26) for $P_n^* \equiv P_s^* \equiv \equiv P_3^* = 0$ and $\sigma_{33}^{*\circ} = 0$, after a number of manipulations we obtain a characteristic equation in the form (3.2) for $i, j = 1, 2$. We represent the elements of the determinant as

$$\alpha_{11}(\zeta_2) = \frac{1}{\zeta_2} \left[a_{33} \frac{a_{11} + \sigma_{11}^{*\circ} \lambda_1^{-2}}{a_{11} + G_{13}} \left(\frac{G_{13}}{a_{11} + \sigma_{11}^{*\circ} \lambda_1^{-2}} - \zeta_2^2 \right) + a_{13}\zeta_2^2 \right] \operatorname{sh} \frac{\varkappa_k}{\zeta_2} \frac{h}{R} \tag{3.11}$$

$$\alpha_{21}(\zeta_2) = \left[\frac{a_{11} + \sigma_{11}^{*\circ} \lambda_1^{-2}}{a_{13} + G_{13}} \left(\frac{G_{13}}{a_{11} + \sigma_{11}^{*\circ} \lambda_1^{-2}} - \zeta_2^2 \right) - 1 \right] \operatorname{ch} \frac{\varkappa_k}{\zeta_2} \frac{h}{R}$$

$$\alpha_{12} \equiv \alpha_{11}(\zeta_3), \qquad \alpha_{22} \equiv \alpha_{21}(\zeta_3)$$

It was assumed that $\zeta_2 \zeta_3 \neq 0$ in deriving the characteristic equation.

Let us examine the stability of a hinge-supported rectangular plate of thickness $2h$, length a and width b under uniform multilateral compression, i.e. for $\sigma_{33}^{*\circ} = 0$. We take the solution of (2.18) in the form

$$\Psi_1 = A_{mn}^{(1)} \operatorname{sh} \frac{\gamma}{\zeta_1} x_3 \cos m \frac{\pi}{a} x_1 \cos n \frac{\pi}{b} x_2, \qquad \gamma = \left[\left(m \frac{\pi}{a} \right)^2 + \left(n \frac{\pi}{b} \right)^2 \right]^{1/2}$$

$$\Psi_i = A_{mn}^{(i)} \operatorname{ch} \frac{\gamma}{\zeta_i} x_3 \sin m \frac{\pi}{a} x_1 \sin n \frac{\pi}{b} x_2, \qquad i = 2,3 \tag{3.12}$$

After a number of manipulations a characteristic determinant can be obtained whose elements have the form (3.11) if the $\varkappa_k/R$ therein are replaced by γ.

Within the scope of plane strain, let us examine the stability of a hinge-supported strip of thickness $2h$ and length l under compression along the Ox_3-axis in the x_1x_3 plane. We select the solution of (2.29) in the form

$$\Psi_2 + \Psi_3 = \left(A \operatorname{ch} \frac{\pi}{l} \zeta_2 x_1 + B \operatorname{ch} \frac{\pi}{l} \zeta_3 x_1\right) \sin \frac{\pi}{l} x_3 \tag{3.13}$$

Substituting (2.27), (2.28), (3.13) in the boundary conditions (2.30) for $P_1^* \equiv P_3^* = 0$, we obtain a characteristic equation (3.2) as a result of the customary procedure, for $i, j = 1, 2$. The determinant elements are

$$\alpha_{11}(\zeta_2) = \zeta_2 \left[\zeta_2^2 - \frac{a_{11}(a_{33} + \sigma_{33}^{*\circ} \lambda_3^{-2}) - a_{13}(a_{13} + G_{13})}{a_{11} G_{13}} \right] \operatorname{sh} \alpha\zeta_2, \qquad \alpha = \pi \frac{h}{l} \tag{3.14}$$

$$\alpha_{21}(\zeta_2) = \left(\zeta_2^2 + \frac{a_{33} + \sigma_{33}^{*\circ} \lambda_3^{-2}}{a_{13}} \right) \operatorname{ch} \alpha\zeta_2, \ \alpha_{22} \equiv \alpha_{21}(\zeta_3), \qquad \alpha_{12} \equiv \alpha_{11}(\zeta_3)$$

Let us consider the surface instability of the lower $x_1 < 0$ half-plane within the scope of plane strain under compression along the Ox_3-axis. The concept of surface instability was apparently first introduced in [22], where the surface instability of a half-plane for an incompressible material was investigated. We take the solution of (2.29) in the form

$$\Psi_2 + \Psi_3 = (A \exp \frac{\pi}{l} \zeta_2 x_1 + B \exp \frac{\pi}{l} \zeta_3 x_1) \sin \frac{\pi}{l} x_3 \tag{3.15}$$

As a result of the customary procedure we obtain the characteristic equation in the form (3.2) for $i, j = 1, 2$. The determinant elements are

$$\alpha_{11}(\zeta_2) = \zeta_2 \left[\zeta_2^2 - \frac{a_{11}(a_{33} + \sigma_{33}^{*\circ}\lambda_3^{-2}) - a_{13}(a_{13} + G_{13})}{a_{11}G_{13}} \right], \quad \alpha_{12} \equiv \alpha_{11}(\zeta_3)$$

$$\alpha_{21}(\zeta_2) = \zeta_2^2 + \frac{a_{33} + \sigma_{33}^{*\circ}\lambda_3^{-2}}{a_{13}}, \qquad \alpha_{22} \equiv \alpha_{21}(\zeta_3) \tag{3.16}$$

Here l is determined by minimizing the roots of the characteristic equation. If $\mathrm{ch}\,\alpha\zeta_i$ and $\mathrm{sh}\,\alpha\zeta_i$ are interchanged in (3.14), and it is considered that $\sigma_{33}^{*\circ} > 0$, we then obtain the characteristic equation for the problem of the formation of a neck in a plane sample under tension.

Let us consider the internal instability under compression along the Ox_3-axis, i.e. for $\sigma_{11}^{*\circ} = 0$ and $\sigma_{33}^{*\circ} < 0$. The concept of internal instability was apparently first introduced in [23], where this phenomenon was examined also for plane strain. Internal instability is understood to be the phenomenon when the solution of the hyperbolic equation appears. The appropriate value of the loading parameter at which the mentioned phenomenon is obtained, is called the critical value. From (2.18) and (2.19) we find that the critical value of the loading parameter is determined as the least root of the equation,

$$G_{13} + \sigma_{33}^{*\circ} \lambda_1^{-2} = 0 \tag{3.17}$$

It must be noted that the quantities $\sigma_{33}^{*\circ}$ and $\sigma_{11}^{*\circ}$ are negative in all the characteristic equations, with the exception of the cases when a neck is formed in plane and cylindrical samples under tension.

All the results of this section have been obtained for an arbitrary form of the potential, and only now must the form of the elastic potential be made specific in order to obtain numerical results.

The form of the characteristic equations is simplified considerably for thin-walled structures if expansions of special functions in the thin-walledness parameter are used. We hence obtain algebraic equations in place of transcendental ones.

4. Elastic potential forms. From (1.6) and the relationships proposed in [19, 24], a correspondence can be established between the quantities ψ_0, ψ_1, ψ_2 and K^*, G^*, ω^* [19, 24]; hence, the quantities a_{ij} and G_{ij} of (2.19) can be expressed in terms of K^*, G^* and ω^* and their derivatives. Therefore, in the presence of experimentally verified dependences of the quantities K^*, G^* and ω^* on the invariants of the strain tensor for a definite material, then numerical values of the critical parameters can be obtained from the characterisitic equations of the preceeding section. Taking account of the notation in [19], let us express the values of the quantities ψ_0, ψ_1 and ψ_2 in terms of values of the quantities K^*, G^* and ω^* and the strain tensor invariants e, e° and φ introduced in [24]

$$\psi_0 = \frac{\partial \Phi}{\partial I_1} = K^* e + 2G^* \left\{ -\frac{\cos(3\varphi + \omega^*)}{3\cos 3\varphi} e + \frac{\sqrt{3}}{e^\circ} \frac{\sin \omega^*}{\cos 3\varphi} \left[\frac{2}{3}(e^\circ)^2 - \frac{1}{9} e^2 \right] \right\}$$

$$\psi_1 = 2 \frac{\partial \Phi}{\partial I_2} = 2G^* \left\{ \frac{\cos(3\varphi + \omega^*)}{3\cos 3\varphi} + \frac{2}{\sqrt{3}} \frac{e}{e^\circ} \frac{\sin \omega^*}{\sin 3\varphi} \right\} \tag{4.1}$$

$$\psi_2 = 3\,\frac{\partial \Phi}{\partial I_3} = 2G^*\left\{-\frac{\sqrt{3}}{e^\circ}\,\frac{\sin\omega^*}{\cos 3\varphi}\right\} \qquad \text{(cont.)}$$

Substituting the partial derivatives of the potential (4.1) into (2.9), we obtain values of the quantities a_{ij} and G_{ij}, expressed in terms of the quantities K^*, G^*, ω^*, e, e°, φ. It must be noted that

$$e \equiv I_1, \qquad e^{\circ 2} = {}^1/_2 I_2 - {}^1/_6 I_1{}^2 \tag{4.2}$$

The relationships (4.1) are simplified considerably if the deviators are similar, i.e. the phase of deviator similarity is $\omega^* = 0$. In this case we obtain from (4.1)

$$\frac{\partial \Phi}{\partial I_1} = \left(K^* - \frac{2}{3}G^*\right)e, \qquad \frac{\partial \Phi}{\partial I_2} = \frac{1}{3}G^*, \qquad \frac{\partial \Phi}{\partial I_3} = 0 \tag{4.3}$$

Therefore, (4.1)–(4.3) afford a possibility of obtaining numerical results from the characteristic equations of the preceding Section when the connection between the components of the generalized stress tensor and the finite strain tensor is given in the form of [24].

A detailed discussion of the elastic potential forms for incompressible rubberlike materials can be found in [2, 25], where recent results can be found in the latter.

The elastic potential is selected in [3, 14, 9, 15, 16] in a form which outwardly agrees with the form of the elastic potential for small deformations, and naturally goes over into it for small deformations. No experimental verification of this kind of elastic potential has been performed for any material in the mentioned papers. In the notation taken herein, this potential can be represented as

$$\Phi = \mu I_2 + {}^1/_2 I_1{}^2, \qquad \lambda,\ \mu \text{ — (Lame parameters)} \tag{4.4}$$

A material with the potential in the form (4.4) is called "semilinear" in [14], and "harmonic" in [9, 15, 16], in relation to the singularities in the solution of the boundary value problems for such a material.

Let us write the values of some quantities obtained from the potential (4.4). From (1.6) and (4.4) we deduce

$$\psi_0 = \lambda I_1, \qquad \psi_1 = 2\mu, \qquad \psi_2 = 0 \tag{4.5}$$

From (2.9) and (4.4) we obtain

$$a_{ij} = \lambda + 2\mu\delta_{ij}, \qquad G_{ij} = \mu \tag{4.6}$$

Therefore, the relationships (2.8) and (2.9) seem to agree with the relationships for small deformations of an isotropic body.

For the three-dimensional problem we deduce from (2.3), (2.4) and (4.4)

$$\sigma_{11}^{*\circ} = {}^1/_2\lambda(\lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2 - 3) + \mu(\lambda_1{}^2 - 1), \quad \sigma_{22}^{*\circ} = {}^1/_2\,\lambda(\lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2 - 3) + {} + \mu\,(\lambda_2{}^2 - 1)$$

$$\sigma_{33}^{*\circ} = {}^1/_2\lambda(\lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2 - 3) + \mu(\lambda_3{}^2 - 1) \tag{4.7}$$

and in the x_1x_3 plane for the plane problem

$$\sigma_{11}^{*\circ} = {}^1/_2(\lambda_1{}^2 + \lambda_3{}^2 - 2)\lambda + \mu(\lambda_1{}^2 - 1), \quad \sigma_{33}^{*\circ} = {}^1/_2(\lambda_1{}^2 + \lambda_3{}^2 - 2)\lambda + \mu(\lambda_3{}^2 - 1) \tag{4.8}$$

Note. The investigation of the characteristic equations of the preceding Section is simplified substantially in the case of the potential (4.4) since the quantities a_{ij} and G_{ij} in (4.6) are independent of λ_i and the components of the subcritical state of stress are expressed simply in terms of λ_i in the relationships (4.7) and (4.8). These equations are almost the same as the corresponding equations of the theory of small subcritical deformations. A number of results within the scope of the potential (4.4) has been obtained in [3, 14, 16, 9].

5. Numerical example. Within the scope of the potential (4.4), let us investigate the stability of a strip of thickness $2h$ and length l under compression along the ox_3-axis. Let us examine the case of plane strain in the x_1x_3 plane, and let us compare it with the solution of this same problem within the scope of the theory of small subcritical deformations [26]. In conformity with (4.3), we obtain for $\sigma_{11}^{*\circ} = 0$

$$\lambda_3^2 = 2E^{-1}(1-\nu^2)\sigma_{33}^{*\circ} + 1, \ \lambda_1^2 = -2\nu(1+\nu)\, E^{-1}\, \sigma_{33}^{*\circ} + 1 \tag{5.1}$$

Substituting (4.6), (5.1) into (3.14), we obtain the characteristic equation to determine $\sigma_{23}^{*\circ} = -p^*$. To accuracy of $\alpha^4 = (\pi h/l)^4$ we obtain

$$p^* \approx p_0\left(1 - \frac{17+3\nu}{1-\nu}\frac{\alpha^2}{15}\right), \qquad p_0 = \frac{E}{1-\nu^2}\frac{\alpha^2}{3} \tag{5.2}$$

Let p denote the intensity of the surface loading referred to the area of the body before deformation. According to (2.14), (5.1) and (5.2) to the same accuracy

$$p \approx p_0\left(1 - \frac{12+8\nu}{1-\nu}\frac{\alpha^2}{15}\right) \tag{5.3}$$

Neglecting deformations as compared with the angles of rotation [4], we obtain by the theory of small subcritical deformations [26]

$$p \approx p_0\left(1 - \frac{12-2\nu}{1-\nu}\frac{\alpha^2}{15}\right) \tag{5.4}$$

Therefore, both by the theory of large subcritical deformations within the scope of the potential (4.4), and by the theory of small subcritical deformations, the value of the critical loading is less than p_0 but their magnitude is different. It must be noted that the magnitude of the critical loading depends essentially on the form of the elastic potential.

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